Dido: Audit of the Hypotheses and Possible Broader Solutions

Working research note, 11 October 2026
Prepared from the repository's original paper and its exotic-surface sequel

The user's proposed direction is that removing the papers' hypotheses may make a broader solution impossible except through numerical methods. This audit separates several meanings of "solution" so that an impossibility claim has definite quantifiers.

A solution could mean: a closed expression for a curve; an implicit differential or variational characterization; an existence and regularity theorem; a classification of candidates; a certified approximation; or a procedure that correctly reports a finite optimizer, an unattained supremum, or an infinite supremum. These are different mathematical targets.

1 Conditions, one at a time

The rows below combine the conditions in the original paper's Section 2, Theorem 4, and Conjecture 5, with the additional analytic program in Section 5.2. Numerical discretization gates are considered separately.

Condition in the papers	**What happens when it fails**	**A broader formulation that accounts for the case**	**Consequence for an impossibility claim**
Smooth attainment of a fixed-length area maximum (Theorem 1)	There may be no maximizing curve. On the cigar, the fixed-length area supremum is infinite. Even the round sphere permits a finite unattained supremum in the unrestricted class of selected disks at exact perimeter.	State the supremum and attainment separately. Allow an answer of "unbounded" or "finite but unattained." Use the fixed-area isoperimetric profile when that is the intended problem.	A universal procedure that must always return a maximizing curve is impossible. Analytic descriptions of nonattainment remain possible.
Sufficient boundary regularity to define J	Corners or weak limits may not have curvature derivative in L^2. The classical J functional may be undefined.	Formulate perimeter and area on finite-perimeter sets. Prove regularity where available. Extending J itself requires a specified relaxation and a separate theorem.	The smooth J theorem cannot simply be asserted for all weak boundaries. Weak variational analysis is an analytic technique, not a numerical workaround.
Embeddedness and an admissible region boundary	Limits may self-touch or become multiple covers; an essential or one-sided curve may not bound the required region.	Use regions as the state, rather than arbitrary parametrized loops. Relax to finite-perimeter sets if topology change is permitted; otherwise prove that the selected topological class is closed under the chosen convergence.	Enlarging the class changes the problem. A solver must distinguish a valid region from a geometric curve with small J.

Condition in the papers	**What happens when it fails**	**A broader formulation that accounts for the case**	**Consequence for an impossibility claim**
Selected side, orientation/coorientation, topology, and number of components	One geometric boundary can enclose different selected areas. A one-sided core on the Klein bottle is inadmissible as a region boundary.	Specify the selected set, use the orientation double cover where useful, and compare separately defined topological classes.	Missing problem data are an information obstruction, not a limitation peculiar to symbolic mathematics.
No escape to infinity / suitable ambient compactness	Cigar boundaries have exact length 2π, area tending to infinity, and J tending to zero.	Prove tightness from geometric restrictions, work in a specified bounded region, or describe escape and nonattainment explicitly. Some noncompact problems still have exact solutions: on the cusp's radial-graph class, $A \leq L$, with equality at a horocycle.	Noncompactness alone does not imply numerical-only solvability. Small J alone cannot supply no-escape control.
Preservation of the equality constraint $L = L_0$ under limits	Perimeter can drop under weak convergence. Compactness of the ambient surface does not repair this.	Prove strong convergence or saturation, or solve a carefully stated relaxed/dual problem and prove its relation to the equality problem. A $\leq$-perimeter formulation may be trivial if the whole compact surface is admissible.	The exact-perimeter problem and the fixed-area problem cannot be interchanged without a theorem.
Regular multiplier: $\kappa^* \neq 0$ for the local quantitative theorem	At a geodesic, the first derivative of length vanishes; the length set is not described by the same regular codimension-one chart.	Use a singular-constraint analysis beginning at higher order; exploit exact geometry in special cases. The Klein-bottle separator family is an explicit analytic singular example.	The attained optimizer's $J = 0$ implication survives. The regular chart and its quantitative estimates require replacement.
A fixed normal-graph gauge, suitable $W^{3,2}$ topology, and a symmetry slice	Reparametrization and isometries introduce artificial kernels; mixed measures or changing coordinates can invalidate an inverse argument.	Fix coordinates, quotient genuine symmetries, and compare norms and densities in a proved chart.	These are requirements for a valid proof, rather than evidence that all exact analysis fails.
Invertibility of the projected Jacobi/curvature operator (Theorem 4)	Kernel directions invalidate the stated inverse estimate and may change the exponent.	Apply Lyapunov–Schmidt reduction when its Fredholm hypotheses hold, analyze the finite-dimensional reduced problem, or use an analytic error-bound theorem with all its hypotheses checked.	The paper's particular linear J estimate does not automatically survive degeneracy. A failure of that estimate is weaker than impossibility of solving the problem.

Condition in the papers	What happens when it fails	A broader formulation that accounts for the case	Consequence for an impossibility claim
A compact smooth zero-set stratum Z_0 (Conjecture 5)	Zero-energy candidates may bifurcate, intersect, have different dimensions, or fail to form a single smooth family.	Analyze strata and bifurcations separately in classes where such structure can be proved. Retain a set-valued candidate description in greater generality.	The proposed global normal-bundle argument cannot be used unchanged. No general classification is supplied merely by writing $J = 0$.
Kernel equals the tangent space of Z_0, with uniform Morse–Bott estimates	Extra kernel directions or spectral gaps tending to zero make local constants nonuniform.	Establish branch-dependent estimates, a uniform spectral gap under added assumptions, or higher-order estimates.	Pointwise stability does not establish a uniform global rate.
Small-J sequential compactness in $W^{3,2}$, with embeddedness and topology retained	A small residual need not place the boundary in any admissible local chart. Self-contact, covering, escape, or degeneration can defeat localization.	Prove this compactness independently or enlarge the compactification and state what its boundary objects mean.	Local inverse estimates do not prove the missing global compactness bridge.
Global comparison among all relevant components of $J = 0$	Distinct strictly stable candidates can have identical boundary neighborhoods and different areas.	Use the global metric and selected-region areas to compare candidates; J remains a necessary-condition diagnostic.	There is a rigorous impossibility theorem for global area selection from boundary-local data alone. See the companion proof note.
Analyticity, Fredholm structure, and norm comparisons for the proposed Łojasiewicz–Simon program	A merely smooth residual may vanish to infinite order; the proposed analytic theorem may not apply.	Impose and verify an analytic setting, or determine a metric-specific modulus by another argument.	Smoothness alone does not provide a universal power law. Realizing a particular flat residual in Dido geometry is a separate construction, not established here.

These conditions protect different claims. Failure of a condition invalidates the proof that uses it; it does not automatically falsify every possible broader statement.

2 Three analytic descriptions of the "degenerate" cases

2.1 Cusp: an exact restricted-class answer

For a periodic graph $r = f(\theta)$ in the end $g = dr^2 + e^{-2r}\,d\theta^2$, selecting the tail gives

$$A = \int_0^{2\pi} e^{-f}\,d\theta, \qquad L = \int_0^{2\pi} \sqrt{f'^2 + e^{-2f}}\,d\theta \geq A.$$

Equality holds exactly when f is constant. For $0 < L \leq 2\pi$, the constant $f = \log(2\pi/L)$ is admissible and has $A = L$. This proves the optimum in the entire radial-graph class without an optimizer run. It does not classify every possible region on a completed surface containing the cusp.

2.2 Klein bottle: a continuous exact family

In the repository's flat quotient, let $0 < h < 3$. The horizontal lifts at heights h and $6 - h$ form one two-sided separator downstairs. Its selected Möbius band has

$$L = 4, \qquad \kappa_g = 0, \qquad J = 0, \qquad A = 4h.$$

Thus A runs through $(0, 12)$. In this selected class the supremum is 12 and is not attained by a proper smooth region with nonempty boundary. The endpoint is a one-sided degeneration. These formulas are an analytic answer describing the obstruction. The experimentally compared disk/separator envelope is a separate restricted-class calculation, not a proof of the full Klein-bottle isoperimetric profile.

2.3 Cigar: an exact nonattainment answer

The companion proof note constructs smooth disks of exact length 2π with unbounded area and $J \to 0$. The correct answer to that fixed-length maximization problem is an infinite supremum and no finite-area maximizing disk. Numerical methods cannot return a maximizing curve for this class.

3 A genuine broader analytic formulation

On a closed smooth connected Riemannian surface, define the fixed-area isoperimetric profile

$$I_M(v) = \inf\{\mathrm{Per}(E) : \mathrm{Area}(E) = v,\ E \text{ a finite-perimeter set}\}, \qquad 0 < v < \mathrm{Area}(M).$$

A minimizing sequence has bounded perimeter. BV compactness gives an L^1-convergent subsequence; area is preserved under this convergence; lower semicontinuity of perimeter produces a minimizer. Standard two-dimensional isoperimetric regularity gives smooth boundary for this unrestricted fixed-area problem. This framework accommodates disconnected regions and nonorientability, and does not start by assuming the paper's disk normal-graph chart.

This is a broader existence/variational solution. It is not an explicit formula for every metric. Nor does it prove attainment in an exact-length, fixed-topology class. Passing between the formulations needs additional branch, admissibility, and saturation arguments. Numerical computation can approximate such a profile, but its validity also requires suitable representation, convergence, and certification assumptions.

The fixed-area constraint also gives a concrete way to handle the geodesic multiplier case. Its derivative is $\delta A = \int \phi \, ds$ and is regular for a nonempty smooth boundary. The multiplier equation $\delta P = \lambda \delta A$ therefore gives $\kappa_g = \lambda$, including $\lambda = 0$, without dividing by κ_g. This avoids the specific singularity of the length-constraint chart. It does not prove that the original exact-length class is a regular manifold at a geodesic.

4 What numerical methods must still assume

A finite Fourier ansatz may miss another topology or another branch. A local optimizer may select a nonglobal stable solution. A finite mesh can miss narrow curvature layers or an interior metric feature. A failed optimizer run is inconclusive. An escaping sequence is not an attained optimizer.

Certified approximation needs an input model and quantitative controls: for example computable metric data, error bounds, a compact search region, regularity/resolution bounds, and treatment of all admissible classes. The finite-query lemma in the companion note shows why arbitrary smooth metrics cannot be integrated exactly from finitely many point samples without further information.

5 The defensible research target

The strongest proved direction from the current material is:

Global Dido area ranking is not determined by boundary-local geometry, even with fixed length, fixed selected disk topology, compact smooth ambient geometry, and strict local stability.

A stronger statement such as "no nonnumerical technique can solve the general problem" requires a precisely specified input class, allowed operations, and required output. The present counterexamples do not establish that statement. They instead identify which global information a proposed solution must use.

The next research choices are therefore precise: a theorem excluding a specified class of boundary formulas; a geometric construction defeating a proposed uniform rate; or a computability/complexity result under a specified representation of the full metric. Failure to find a formula is not evidence of such a theorem.

Sources and verification

- Original paper: `/workspace/Dido/dido_merged_2026-09-08.pdf`, Sections 2, 5, and 5.2.
- Sequel: `/workspace/Dido/dido_relaxation_exotic_surfaces.pdf`, Sections 6–10.
- Full-source witness: `/workspace/dido-dev/dido_exotic_surfaces_reproducibility/dido_argument_audit`
- Companion: `impossibility-note.md` in this directory.
- The existing argument audit completed with all 19 checks passing. Its logical entries are explanatory encodings, and its sampled entries are crosschecks; the analytic proofs in the companion note carry the mathematical claims. No proof-assistant verification or priority claim is made.