Dido's Problem: Technical Overview

T. Schaefer

October 2026

1 The problem

Here "manifold" means a smooth two-dimensional Riemannian surface (M, g); its metric determines lengths and areas. Given a prescribed boundary length L_0 and an admissible class $\mathcal{A}$ of selected regions, Dido's problem asks for

$$\sup_{D\in\mathcal{A},\ \mathrm{Length}_g(\partial D)=L_0} \mathrm{Area}_g(D),$$

and for the regions attaining that supremum, if any. The admissible class matters: disks, annuli, disconnected regions, and opposite sides of the same boundary can give different answers.

The familiar companion problem minimizes perimeter at prescribed area. Its solutions define the isoperimetric profile. Passing between the two formulations requires appropriate admissibility and branch conditions; exact perimeter is not automatically preserved under limiting operations.[1]

2 Historical approaches

Classical symmetry methods, notably Steiner's nineteenth-century symmetrization, established the circle's privileged role in the plane. Constant-curvature geometry yields corresponding results for geodesic disks in the sphere and hyperbolic plane, with the appropriate selected-area interpretation. Early symmetry arguments also exposed the need to establish existence.[2]

Variational methods identify the equilibrium condition: an optimal smooth boundary has constant geodesic curvature. Twentieth-century geometric measure theory supplied a deeper foundation through finite-perimeter sets, compactness, and regularity. On closed smooth surfaces, these methods establish existence for the unrestricted fixed-area perimeter problem, without first guessing the boundary's shape.

Geometric classification then exploits special structure. Morgan, Hutchings, and Howards studied surfaces of revolution with decreasing Gaussian curvature; Ritoré analyzed constant-geodesic-curvature curves and isoperimetric domains in rotationally symmetric surfaces. Symmetry and curvature assumptions make otherwise difficult global comparisons tractable.[3]

A complementary tradition asks how far an almost-optimal region can depart from an optimizer. Bonnesen-type inequalities and modern Euclidean quantitative isoperimetry relate geometric asymmetry to isoperimetric deficit.[4] Together, these approaches address equilibrium, existence, classification, and stability—distinct parts of the problem.

[1] A. Ros, "The isoperimetric problem," in *Global Theory of Minimal Surfaces*, Clay Mathematics Proceedings 2 (2005), 175–209.

[2] R. Osserman, "The isoperimetric inequality," Bulletin of the American Mathematical Society 84 (1978), 1182–1238.

[3] F. Morgan, M. Hutchings, H. Howards, "The isoperimetric problem on surfaces of revolution of decreasing Gauss curvature," Transactions of the AMS 352 (2000), 4889–4909; M. Ritoré, "Constant geodesic curvature curves and isoperimetric domains in rotationally symmetric surfaces," Communications in Analysis and Geometry 9 (2001), 1093–1138.

[4] R. Osserman, "Bonnesen-style isoperimetric inequalities," American Mathematical Monthly 86 (1979), 1–29;

3 Our approach and its success

We introduced curvature variation as a companion functional:

$$J[\gamma] = \int_\gamma \left(\frac{d\kappa_g}{ds}\right)^2 ds.$$

For several boundary components, the energies are summed. It measures variation of intrinsic bending; it is not the full physical jerk.

The first success is exact and conditional: every smoothly attained fixed-length area maximizer has constant κ_g, hence $J = 0$. The constant-curvature characterization is classical. Our paper applies it to the least-variation conjecture and supplies a local quantitative estimate

$$A(\gamma_*) - A(\gamma) \le C\, J[\gamma]$$

near a regular optimizer whose projected linearized curvature operator is invertible.[5]

The computational campaign tests these ideas across variable-curvature metrics, tori, the Klein bottle, a cusp, a paraboloid, a horn, and Hamilton's cigar. It provides reproducible geometry checks and numerical evidence within declared curve families.[6]

The decisive global obstruction is analytic: one smooth compact sphere contains two strictly stable, equal-length disk boundaries with $J = 0$, yet their areas differ by 8π. Their selected-side boundary neighborhoods are even isometric. Consequently local boundary geometry and local stability cannot universally rank enclosed areas. Our current proof draft makes that information obstruction explicit.[7]

4 What escapes the approach

Attainment can fail. Cigar boundaries can retain exact length while enclosed area diverges and $J \to 0$. Compactness alone also fails to protect the unrestricted exact-perimeter problem: on a round sphere, increasingly oscillatory boundaries can surround shrinking disks while their selected complements approach the whole sphere's area without attaining it.

Admissibility can fail. Curves can self-touch, approach multiple covers, change topology, or degenerate into one-sided curves that do not bound the required region. Perimeter can drop in a weak limit.

Local estimates can fail. Geodesics make the length constraint singular; additional Jacobi kernels, bifurcations, and vanishing spectral gaps defeat the stated inverse estimate or require different rates.

Global selection remains separate. Small J can occur near an unstable equilibrium or a stable but inferior one. Across multiple components, J also misses disagreements between their constant curvatures. None of these failures disproves the attained-optimizer implication. They limit what can be inferred from its diagnostic.

N. Fusco, F. Maggi, A. Pratelli, "The sharp quantitative isoperimetric inequality," Annals of Mathematics 168 (2008), 941–980.

[5] T. Schaefer, *Dido's Problem and the Curvature-Variation Energy*, 8 September 2026, Sections 2 and 5. The local theorem uses a fixed $W^{3,2}$ normal-graph gauge and symmetry reduction.

[6] GPT/OpenAI and T. Schaefer, *Dido Relaxation Across Classical Exotic Surfaces*, corrected edition, 9 September 2026; accompanying reproducibility bundle.

[7] The sphere witness is in the preceding paper, Section 8. The collar formulation and explicit round-sphere sequence are developed in the current working proof note (`/workspace/dido-research/impossibility-note.md`), pending independent mathematical review.

5 Approaches still available

A global isoperimetric-profile theory can investigate existence and compare area branches, using the entire metric and explicitly stated region classes. Curvature comparison, symmetry, topological classification, and global variational inequalities can establish additional special cases.

Degenerate equilibria may admit bifurcation analysis or Lyapunov–Schmidt reduction. In suitable analytic settings, Łojasiewicz–Simon methods may provide local error bounds, once their analytic, Fredholm, and norm hypotheses are verified.[8] Such bounds still need a separate global selection argument.

Certified numerical methods could combine controlled discretization, rigorous bounds, and exploration of competing topologies. They require quantitative input and error assumptions; numerical approximation cannot produce a nonexistent maximizer.

A successful extension must supply global information that J omits, establish the relevant compactness, and distinguish an attained optimum from a finite unattained or infinite supremum. Our work excludes a universal selector based solely on local boundary data. Broader analytic and computational theories remain possible.

[8] L. Simon, "Asymptotics for a class of non-linear evolution equations, with applications to geometric problems," Annals of Mathematics 118 (1983), 525–571.