CONCEPT NOTE

From Dido's Boundary to Experimental Design

Abstract surfaces, information perimeters, and equilibrated trial points

> Thomas Schaefer: Dear GPT: Do you agree that the idea of a 'surface' and 'circumference' can be abstracted to certain data sets that have no direct relationship to geometry?

Yes - decisively - provided that we preserve the structural roles rather than the visual meanings.

A surface need not be a sheet floating in three-dimensional space. For applied mathematics, it may be any collection of possible states equipped with locality: a rule telling us which states are near, which changes are permitted, and what it costs to cross from one regime to another. Circumference then ceases to mean a tape-measured distance. It becomes the cost, complexity, exposure, or information flow across the boundary of a selected subset.

This is the essential liberation. Geometry supplies a language of structure. Physical shape is only one interpretation of that language.

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1. The abstraction that matters

Suppose a dataset, graph, probability model, ontology, or experimental factor space supplies four ingredients: a set of states, a notion of neighborhood, a measure of the value contained in a subset, and a cost assigned to its boundary. Then the Dido question survives intact:

$$\text{maximize } \mu(S) \text{ subject to } P(S) = P_0$$

Here S is the selected subset, $\mu(S)$ is its mass, information, value, coverage, or probability, and $P(S)$ is its perimeter in the abstract system. On a weighted graph, for example, $\mu(S)$ can be the sum of node weights inside S, while $P(S)$ is the total weight of edges connecting S to its complement. Nothing in that definition requires physical geometry.

Dido geometry	Design of experiments
Surface or manifold	Factor, state, or information space
Selected region	Domain effectively explored by the experiment
Enclosed area	Information, coverage, or inferential capability
Circumference	Trial budget, transition cost, or exposed interface
Geodesic curvature	Local marginal information pressure
Constant curvature	Equalized sensitivity at supported trial points
J = 0	No local variation in marginal value along the support

The important move is not to call everything a surface. It is to prove that the proposed abstraction possesses the operations that made the geometric theorem work: local variation, a boundary measure, an interior measure, and a meaningful constraint class.

> Structural principle. Preserve adjacency, variation, measure, boundary, and admissibility. The picture may disappear; the mathematics does not.

2. The bridge to Design of Experiments

In Design of Experiments (DOE), a design is a finite distribution of trials over a factor space. For regression functions $f(x)$, trial points x_i, and allocation weights w_i, its information matrix is

$$M(\xi) = \Sigma_i\, w_i\, f(x_i)\, f(x_i)^T.$$

A D-optimal design maximizes $\log \det M(\xi)$, equivalently enlarging the information volume available for parameter estimation. The sensitivity function

$$\psi_\xi(x) = f(x)^T\, M(\xi)^{-1}\, f(x)$$

measures the marginal informational pressure at a possible trial point. The classical equivalence theorem says that an optimal design caps this sensitivity everywhere and reaches the cap at every supported trial point. Thus the supported points share a common marginal value.

> The central analogy. Constant geodesic curvature along an optimal Dido boundary and equal sensitivity across the support of an optimal experimental design are both equilibrium conditions.

3. A Dido-style relaxation for trial points

The analogy suggests a computable diagnostic. Connect nearby supported trial points by an adjacency graph E. Let a_{ij} encode their proximity, transition cost, or experimental relationship. Define

$$J_{DOE}(\xi) = \Sigma_{(i,j)\in E}\, a_{ij}\, [\psi_\xi(x_i) - \psi_\xi(x_j)]^2.$$

This functional is nonnegative. It vanishes when adjacent supported trial points have equal marginal information sensitivity. A relaxation algorithm could move points and redistribute weights to reduce JDOE, producing a family of informationally equilibrated candidate designs.

The proposal is not that JDOE replaces D-, A-, E-, G-, or Bayesian optimality. It supplies a search geometry and a falsifiable diagnostic. The established DOE criterion remains the global judge.

A practical two-stage procedure

1. Relax. Move trial locations and allocation weights to reduce variation in marginal information pressure, while enforcing the trial budget and physical constraints.

2. Select. Compare distinct equilibrated candidates using the chosen information criterion, prediction loss, robustness, and cost.

3. Gate. Reject candidates that escape to unrealistic factor limits, exploit singular leverage, omit required operating regimes, or violate transition and safety constraints.

4. Refine. Increase resolution, add candidate points, perturb the design, and verify that both the criterion and the equilibrium diagnostic remain stable.

4. When circumference becomes campaign cost

The analogy becomes especially concrete when trials must be executed in sequence. Changing temperature, pressure, tooling, aircraft configuration, software state, or manufacturing setup may cost more than repeating a trial at the current condition. The experiment is then a path through factor space, and its circumference can be interpreted as total campaign transition cost.

$$\text{maximize information explored subject to fixed path or switching cost}$$

This formulation differs from ordinary fixed-run DOE. It combines design quality with routing and reconfiguration. The optimal trial set and its execution order become coupled. In that setting, Dido's language is not ornamental: it points toward a genuine constrained variational problem.

5. What the exotic surfaces warn us about

Our geometric experiments supply a ready-made red team for the abstract DOE proposal. Each manifold isolates a failure mode that has a direct experimental interpretation.

Hamilton's cigar - quiet escape. A sequence can look increasingly equilibrated while migrating toward extreme or inaccessible factor values. Small variation does not guarantee an attained or usable optimum.

Gabriel's horn - numerical boundary layers. Extreme designs can produce sharply localized sensitivity changes. Coarse discretization may report stable information and budget while completely missing the boundary energy. Resolution must track the narrowing scale.

The Klein bottle - hidden topology. Identical local diagnostics can belong to designs covering fundamentally different regimes. The selected side, connectivity, and admissible component are part of the state, not metadata to be added later.

The smooth sphere - several excellent equilibria. Distinct, strictly stable candidates can share zero variation yet have different total value. A local equilibrium certificate cannot choose the global champion.

The torus - homology and disconnected search. A local optimizer can remain trapped in the wrong structural class. Multistart search is not a luxury when the admissible space has qualitatively different components.

> Necessary is not sufficient. Equalized marginal information can identify the equilibrium set. It cannot, by itself, establish existence, admissibility, robustness, or global superiority.

6. Testable research questions

The abstraction earns its keep only if it produces experiments that can fail. The following are concrete starting questions.

Q1. Does minimizing JDOE accelerate convergence to classical D-optimal support points compared with direct determinant optimization?

Q2. Does JDOE reveal multiple equilibrium components that ordinary single-start optimization conceals?

Q3. Under compact factor bounds and nonsingular information, can a local information deficit be bounded by a constant times JDOE near a nondegenerate optimum?

Q4. When transition cost defines the perimeter, do optimal campaigns develop constant marginal information pressure along their executed path?

Q5. Which escape, topology, and stability gates are necessary before a low-JDOE design may be trusted?

7. The proposition

The most defensible conclusion is not that Dido's theorem has already solved Design of Experiments. It is that both fields expose the same optimization architecture:

> A constrained optimum equalizes a local marginal quantity along its active boundary or support.

That equilibrium condition can be converted into a nonnegative variation functional. Relaxation toward its zero set may generate excellent candidates. But global selection requires additional information: the objective value, the admissible component, compactness or escape control, and the topology of what has been selected.

The applied-mathematical opportunity lies precisely here. We can carry the logic of a theorem out of its original picture without pretending that the new domain is physically geometric. The equations are not the prison. Treating their first interpretation as their only interpretation is.

> Working hypothesis. Optimal trial points form an equilibrated informational boundary. A Dido-style relaxation can help locate that boundary; a classical DOE criterion must still decide which equilibrium is globally best.

A note on status

This document proposes a research program. The Dido first-variation result and the classical DOE equivalence theorem are established mathematics. Their structural correspondence is exact at the level described here. The specific JDOE functional, its convergence properties, and its usefulness as an optimizer remain hypotheses to be tested rather than claimed results.

References

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